Qualitative properties of a model of coupled drilling oscillations

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***Abstract*—The model of the axial and torsional vibrations for a drillstring with distributed parameters is obtained using the variational approach within the Hamiltonian Mechanics. Further there is considered the model of the axial vibrations which is a linear system with control and perturbation input signals. To the aforementioned equations of the model a system of equations with deviated argument is associated by integration along the characteristics. The system with deviated argument is of neutral type and allows construction of the basic theory but has its difference operator marginally (critically, not strongly) stable. This aspect is discussed finally and suggests the use of the methodology of the singular perturbations.**

***Index Terms*—Distributed Systems, Linear Systems, Industrial Applications**

I. Introduction. Problem statement

Progress of the oil extracting industry is accompanied by such unpleasant phenomena as vibrations of the drilling equipment. These vibrations turn to be the main cause of equipment breaking which is a constant source of material and financial losses. Consequently the scientific and technical concern for vibration analysis and vibration quenching has become an urgent task. The last years witnessed an increase of research production and research literature on the subject, implying more and more the automatic control. We mention here but a quite recent reference having itself a long list of references [1]. References of two recently accepted PhD theses [2], [3] can complete the image of the recent progress in the field.

The development of research has pointed out two main directions: dynamics modeling and control of vibration quenching. The first direction of research focuses on distributed parameters for the drill and on nonlinear representation for the rock-bit interaction. The second direction of research is control of vibrations. Basically vibration quenching *via* feedback control reduces to the increase of a variable damping factor using the control of the electrical drive. With respect to control one can mention adaptive control [4], H_∞ control [5], methods derived from standard methods [6], standard torque control and other. Worth mentioning that drive analysis for the control can make the phenomena representation even more complex: replacement of the d.c. drives by a.c. induction motors can be a source of hidden oscillations [7].

For these reasons the present paper will focus on distributed parameter modeling as main source of drillstring undamped (or weakly damped) oscillations. Following a quite large stream of research as reflected in the published literature, the drillstring will be considered as having distributed parameters. The model itself will be obtained using the variational approach using the Hamiltonian Mechanics [8] adapted to distributed parameters [9], [10] and the standard Euler Lagrange variations [11].

The model of the coupled torsional and axial vibrations will be obtained and compared to that in common use in physics and engineering [12], [13]. Comments on the significance will accompany each relevant step of the modeling. Next the analysis will focus on the axial vibrations since this model is less studied within the framework of the distributed parameters. Following [14]–[16] and our own methodology [17], [18] there is associated to the initial model a system of coupled delay differential and difference equations; the connection of the two kinds of mathematical models is given by the one to one correspondence between their solutions.

From now on the analysis deals with the aforementioned system of functional equations which turns to be of neutral type. Since it is linear, as the source system of partial differential equations, the characteristic equation is of interest for inherent stability studies. Here a major difficulty occurs: the difference operator is only marginally (not strongly) stable, being thus in a critical case. Consequently the system of neutral equations (and, with him, the initial one) can display some "ugly i.e. unpleasant" phenomena which had been reported long ago [19] but were forgotten after the spread of the strong stability assumption due to J.K. Hale and M. A. Cruz [20]. The fact that such limit cases can occur in applications (and not as mathematical curiosities/pathologies) gives to the entire problem a new perspective. For the given drillstring model an approach based on singular perturbations is suggested. Other open problems are pointed out in the Conclusion section.

II. Modeling through the variational approach

II.1 Kinetic and potential energies. The first step in this approach to modeling is to write down the system energy. The kinetic energies are as follows

$$\mathscr{E}_{k\theta}(t) = \frac{1}{2}\left[J_m\dot{\theta}_m(t)^2 + J_b\theta_t(L,t)^2 + \int_0^L \rho(s)I_p(s)\theta_t(s,t)^2\mathrm{d}s\right] \quad (1)$$

represents the rotation kinetic energy, where the following notations have been made

J_m, J_b - the moment of inertia of the driving mechanism for the rotation motion and of the bit, respectively;

$\theta_m(t)$ - rotation angle of the driving motor;

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$\rho(s)$ - distributed density of the drillstring shaft (space dependent);

$I_p(s)$ - polar momentum of the drillstring shaft - a geometric parameter (space dependent);

$\theta(s,t)$ - rotation angle of the drillstring shaft at time t and space coordinate s, $0 \leq s \leq L$; here and elsewhere L is the length of the drillstring shaft

It will appear later that a certain difference between $\theta_m(t)$ and the rotation angle of the drillstring shaft does exist, accounting for the elastic torsional strain of the drillstring $\nu(s,t)$

$$\nu(s,t) = \theta(s,t) - \theta_m(t) \tag{2}$$

Next

$$\mathscr{E}_{kH}(t) = \frac{1}{2}\left[m_0 \dot{z}_H(t)^2 + m_b z_t(L,t)^2 + \int_0^L \rho(s)\Gamma(s) z_t(s,t)\mathrm{d}s\right] \tag{3}$$

represents the linear motion kinetic energy, where the following notations have been made:

m_0, m_b - the mass of the vertical driving mechanism and of the bit, respectively;

$z(s,t)$ - the shaft linear motion coordinate: we have, as in the rotation case

$$z(s,t) = \zeta(s,t) + z_H(t) \tag{4}$$

with $\zeta(s,t)$ - the linear elastic strain;

$\Gamma(s)$ - the cross section area of the shaft.

The potential energies accumulated through shaft deformation are as follows

$$\mathscr{E}_{p\theta}(t) = \frac{1}{2}\int_0^L G(s)I_p(s)\nu_s(s,t)^2\mathrm{d}s = \int_0^L G(s)I_p(s)\theta_s(s,t)^2\mathrm{d}s \tag{5}$$

where $G(s)$ is the shear modulus, represents the potential energy associated with torsion. Next

$$\mathscr{E}_{pH}(t) = \frac{1}{2}\int_0^L E(s)\Gamma(s)\zeta_s(s,t)^2\mathrm{d}s = \int_0^L E(s)\Gamma(s) z_s(s,t)^2\mathrm{d}s \tag{6}$$

represents the potential energy; here $E(s)$ is the Young elasticity modulus, associated with compression/stretching. An additional mention for the notations: for the distributed variables i.e. $\theta(s,t)$, $z(s,t)$ etc, the indices t, s denote partial derivation with respect to that variable and repeated indices mean second order or mixed partial derivation.

Here one has to mention that according to the form of the potential energies, one type of shaft or another may result (Bernoulli, Timoshenko etc - see [21]). For (5) and (6) the string model is obtained i.e. the simplest case of distributed parameter shaft modeling.

II.2 Forces and momenta. The next step is to write down the active momenta and forces. For rotation we shall have the following momenta

$\tau_a(t)$ - the active momentum that rotates the shaft - it can result as an output of the driving mechanism;

$\tau_o(t)$ - the damping momentum at motor's shaft; we *assume* that $\tau_o(t) = -\gamma_o'\dot{\theta}_m(t)$;

$\tau_1(t)$ - the momentum that is transmitted to the load; we *assume* that $\tau_1(t) = \gamma_l'\dot{\theta}_m(t)$;

$\tau_2(t)$ - the load momentum of the shaft; we *assume* that $\tau_2(t) = -\gamma_l'\theta_t(0,t)$ (the momenta $\tau_1(t)$ and $\tau_2(t)$ appear as virtual momenta when separating the motor shaft from the drilling shaft; these momenta would have been equal for perfect rigidity - zero elastic strain);

$\tau_d(s,t)$ - the distributed friction momentum: $\tau_d(s,t) = -\gamma_\theta(s)I_p(s)\theta_t(s,t)$;

$\tau_b(t)$ - the load momentum at the bit: we *assume*, according to [1], [12], [13], that $\tau_b(t) = -T_b(\theta_t(L,t)$ where $T_b(\sigma)$ is one of the functions described in the aforementioned references.

Except the load momentum, all dependencies are assumed linear with the proportionality coefficients γ_o', γ_1' etc; their physical dimension is thus clear.

Using the momenta thus defined we are in position to write down the work of the momenta in the rotation motion

$$\begin{aligned}\mathscr{W}_{m\theta}(t) = (\tau_a(t) + \tau_o(t) + \tau_2(t))\theta_m(t) + \tau_1(t)\theta(0,t) + \\ + \int_0^L \tau_d(s,t)\theta(s,t)\mathrm{d}s + \tau_b(t)\theta(L,t)\end{aligned} \tag{7}$$

Note that the expressions of the momenta are to be substituted in $\mathscr{W}_{m\theta}(t)$ but only after applying the variational principles: for the work there is applied the principle of the virtual displacements - only the generalized coordinates are subject to the variations (not the forces/momenta) [8].

For the vertical motion we shall have the following forces which are involved in the work

$f_H(t)$ - the active force that regulates ground penetration; it can result as an output of the brake motor incorporated in the driving mechanism;

$f_o(t)$ - the damping force within the driving mechanism; we *assume* that $f_o(t) = -\gamma''_o \dot{z}_H(t)$;

$f_1(t)$ - the force that is transmitted to the load; we *assume* that $f_1(t) = \gamma''_l \dot{z}_H(t)$;

$f_2(t)$ - the load force $f_2(t) = -\gamma''_l z_t(0,t)$ (here also forces f_1 and f_2 appear as virtual forces when separating the drive from the drillstring; they would have been equal for perfect rigidity - zero elastic strain);

$f_d(s,t)$ - the distribute friction force on the vertical: $f_d(s,t) = -\gamma_H(s)\Gamma(s)z_t(s,t)$;

$f_b(t)$ - the friction force at the bit, induced by the friction momentum and *assumed* to be

$$f_b(t) = -\frac{2}{\mu\kappa R_b}T_b(\theta_t(L,t)) \tag{8}$$

where we denoted: R_b - the bit radius; κ - the conversion coefficient of the rolling friction into sliding friction; μ - a friction coefficient.

Here also, except the load force at the bit, all dependencies are assumed linear with the proportionality coefficients γ''_o, γ''_1 etc; their physical dimension is thus clear.

We are now in position to write down the work of the aforementioned forces

$$\mathscr{W}_{mH}(t) = (f_H(t) + f_o(t) + f_2(t))z_H(t) + f_1(t)z(0,t) + \\ + \int_0^L f_d(s,t)z(s,t)\mathrm{d}s + f_b(t)z(L,t) \tag{9}$$

(Here also the expressions of the forces are to be substituted in $\mathscr{W}_{mH}(t)$ only after applying the variational principles: under the principle of the virtual displacements, only the generalized coordinates are subject to the variations).

II.3 Variations and resulting equations. We write down the complete Hamiltonian, incorporating the work of the external forces and momenta and the functional to be minimized

$$\begin{aligned} I(t_1,t_2) &= I_\theta(t_1,t_2) + I_H(t_1,t_2) = \\ &= \int_{t_1}^{t_2} (\mathscr{E}_{k\theta}(t) - \mathscr{E}_{p\theta}(t) + \mathrm{W}_{m\theta}(t))\mathrm{d}t + \\ &+ \int_{t_1}^{t_2} (\mathscr{E}_{kH}(t) - \mathscr{E}_{pH}(t) + \mathrm{W}_{mH}(t))\mathrm{d}t \end{aligned} \tag{10}$$

This functional is quadratic. Following the standard procedure we consider the Euler Lagrange variations of the generalized coordinates with respect to some trajectory supposed to ensure the minimum of (10) [11]

$$\theta(s,t) = \bar{\theta}(s,t) + \varepsilon\vartheta(s,t) \ ; \ \theta_m(t) = \bar{\theta}_m(t) + \varepsilon\vartheta_m(t) \ ;$$

$$z(s,t) = \bar{z}(s,t) + \varepsilon\zeta(s,t) \ ; \ z_H(t) = \bar{z}_H(st) + \varepsilon\zeta_H(t)$$

We thus obtained the perturbed functional $I^\varepsilon(t_1,t_2)$ which is quadratic in ε. The necessary extremum conditions are obtained from

$$\frac{\mathrm{d}}{\mathrm{d}t} I^\varepsilon(t_1,t_2)\,|_{\varepsilon=0} = 0 \tag{11}$$

Following the methodology of [11], see also [18], the following equations are obtained: a) for the torsional vibrations

$$\begin{aligned} &-\rho(s)I_p(s)\theta_{tt} + (G(s)I_p(s)\theta_s)_s - \gamma_\theta(s)I_p(s)\theta_t = 0 \\ &-J_m\ddot{\theta}_m + \tau_a(t) - \gamma_o'\dot{\theta}_m - \gamma_l'\theta_t(0,t) = 0 \\ &G(0)I_p(0)\theta_s(0,t) + \gamma_l'\dot{\theta}_m(t) = 0 \\ &-J_b\theta_{tt}(L,t) - G(L)I_p(L)\theta_s(L,t) - T_b(\theta_t(L,t)) = 0 \end{aligned} \tag{12}$$

b) for the axial vibrations

$$\begin{aligned} &-\rho(s)\Gamma(s)z_{tt} + (E(s)\Gamma(s)z_s)_s - \gamma_H(s)\Gamma(s)z_t = 0 \\ &-m_o\ddot{z}_H + f_H(t) - \gamma"_o\dot{z}_H - \gamma"_l z_t(0,t) = 0 \\ &E(0)\Gamma(0)z_s(0,t) + \gamma_l''\dot{z}_H(t) = 0 \\ &-m_b z_{tt}(L,t) - E(L)\Gamma(L)z_s(L,t) - T_b(\theta_t(L,t)) = 0 \end{aligned} \tag{13}$$

The similarity of the two sets of equations is obvious. In both cases they define non-standard boundary value problems for hyperbolic partial differential equations. Since we discussed the torsional vibrations in several papers, the present paper will focus on the equations of the axial vibrations.

III. The basic theory for the model of the axial vibrations

III.1 Some remarks on the model. We shall start here from the basic model (13) where the active force $f_H(t)$ is used as a control signal and $f_b(t) = -2(\mu\kappa R_b)^{-1}T(\theta_t(L,t))$ is also an external force (from the system describing the torsional vibrations) and represents the perturbation signal. Overall system (13) describes a structure with distributed parameters - a propagation equation with non-standard boundary conditions - containing ordinary differential equations at the boundaries.

Another observation concerns the state variables $z_H(t)$ and $z(s,t)$: they enter in (13) only by their time and space derivatives - they are *cyclic variables* and the order of the equations can be thus reduced.

The final remark at this level of generality concerns the comparison to the model considered in [13]. By taking $m_o = 0$ in (13) and relying on singular perturbations, this system reduces to

$$\begin{aligned} &-\rho(s)\Gamma(s)z_{tt} + (E(s)\Gamma(s)z_s)_s - \gamma_H(s)\Gamma(s)z_t = 0 \\ &\gamma_o"E(0)\Gamma(0)z_s(0,t) - (\gamma_l'')^2 z_t(0,t) + (\gamma_l'')^2 f_H(t) = 0 \\ &-m_b z_{tt}(L,t) - E(L)\Gamma(L)z_s(L,t) + f_b(t) = 0 \end{aligned} \tag{14}$$

which has the same structure as the model denoted by (17a)-(17b) in [13].

III.2 The symmetric Friedrichs form. We continue the analysis at this level of generality, for a while, by introducing for (13) and (14) the so called symmetric Friedrichs form with the new variables

$$v(s,t) := z_t(s,t) \ , \ w(s,t) := z_s(s,t) \ ; \ v_H(t) := \dot{z}_H(t) \tag{15}$$

(Observe that $v(s,t)$ is a velocity and we introduced also the local velocity $v_H(t) := \dot{z}_H(t)$ at the ground level - taking into account that the state variables are *cyclic*. Consequently system (13) becomes

$$\begin{aligned} &-\rho(s)\Gamma(s)v_t + (E(s)\Gamma(s)w)_s - \gamma_H(s)\Gamma(s)v = 0 \\ &w_t - v_s = 0 \ ; \ E(0)\Gamma(0)z_s(0,t) + \gamma_l''v_H(t) = 0 \\ &-m_o\dot{v}_H - \gamma_o"v_H - \gamma"_l v(0,t) + f_H(t) = 0 \\ &-m_b v_t(L,t) - E(L)\Gamma(L)w(L,t) + f_b(t) = 0 \end{aligned} \tag{16}$$

(We do not reproduce the new form of (14) since it follows from (16) by taking $m_o = 0$ and eliminating v_H between the two equations at $s = 0$).

III.3 Steady states. We discuss next the steady states of (16) which correspond to a constant axial speed and a constant elastic strain; both f_H and f_b have to be assumed as having constant values: f_b will be constant since the steady state angular velocity θ_t is constant and f_H is proportional to some axial velocity supplied by the brake motor of the drive. The steady state equations will be

$$\begin{aligned} &-\gamma_H(s)\bar{v}(s) + (E(s)\Gamma(s)\bar{w}(s))_s = 0 \ , \ \bar{v}_s(s) = 0 \\ &E(0)\Gamma(0)\bar{w}(0) + \gamma"_l\bar{v}_H = 0 \ , \\ &-\gamma"_o\bar{v}_H - \gamma"_l\bar{v}(0) + \bar{f}_H = 0 \\ &-E(L)\Gamma(L)\bar{v}(L) + \bar{f}_b = 0 \end{aligned} \tag{17}$$

and their solution reads as follows

$$\bar{v}(s) \equiv \bar{v} = \frac{1}{E(L)\Gamma(L)}\bar{f}_b \; ; \; \bar{v}_H = \frac{1}{\gamma\text{'}_0}\bar{f}_H - \frac{\gamma\text{'}_l}{\gamma\text{'}_0} \cdot \frac{1}{E(L)\Gamma(L)}\bar{f}_b$$
$$\bar{w}(s) = \frac{\gamma\text{'}_l}{\gamma\text{'}_0 E(s)\Gamma(s)}\left[-\bar{f}_H + \frac{\bar{f}_b}{E(L)\Gamma(L)}\left(\gamma\text{'}_l + \int_0^s \gamma_H(\lambda)\mathrm{d}\lambda\right)\right] \quad (18)$$

III.4 The Riemann invariants and the functional differential equations. From now on we shall follow the basic assumptions of the references dealing with distributed parameters of the drillstring - see [1], [13], also their lists of references. We shall assume homogeneous materials for the drillstring i.e. the parameters E, Γ, ρ, γ_H to be independent of s and also take $\gamma_H \equiv 0$ (negligible) - the consequences of neglecting the internal material damping of the drillstring will be discussed later. Under these circumstances equations (16) take the form

$$\begin{aligned} &\rho v_t - E w_s = 0 \;,\; w_t - v_s = 0 \\ &E\Gamma w(0,t) + \gamma\text{'}_l v_H(t) = 0 \;,\; m_o \dot{v}_H + \gamma\text{'}_o v_H + \gamma\text{'}_l v(0,t) = f_H(t) \\ &m_b v_t(L,t) + E\Gamma v(L,t) = f_b(t) \end{aligned} \quad (19)$$

The basic theory (existence, uniqueness and continuous data dependence) will be constructed for (19) based on a method announced firstly in [17], described later in several papers of the first author and having a quite definite form in [18]; this method was inspired by the papers [14]–[16]. The approach consists in the following steps: 1^o Introduce the Riemann invariants $w^{\pm}(s,t)$ by

$$w^{\pm}(s,t) = v(s,t) \mp c_a w(s,t) \;,\; c_a = \sqrt{E/\rho} \quad (20)$$

and the converse

$$\begin{aligned} v(s,t) &= \frac{1}{2}(w^-(s,t) + w^+(s,t)) \\ w(s,t) &= \frac{1}{2c_a}(w^-(s,t) - w^+(s,t)) \end{aligned} \quad (21)$$

to obtain instead of (19) the system

$$\begin{aligned} &w_t^{\pm} \pm w_s^{\pm} = 0 \\ &\gamma\text{'}_l v_H(t) + \frac{E\Gamma}{2c_a}(w^-(0,t) - w^+(0,t)) = 0 \\ &m_o \dot{v}_H + \gamma\text{'}_o v_H + (\gamma\text{'}_l/2)(w^+(0,t) + w^-(0,t)) = f_H(t) \\ &m_b \frac{\mathrm{d}}{\mathrm{d}t}(w^+(L,t) + w^-(L,t)) + \\ &\qquad + \frac{E\Gamma}{c_a}(w^-(0,t) - w^+(0,t)) = 2f_b(t) \end{aligned} \quad (22)$$

2^o The Riemann invariants are constant along the characteristic lines $t^{\pm}(\sigma;s,t) = t \pm (\sigma - s)/c_a$ - the forward wave $w^+(s,t)$ along $t^+(\sigma;s,t)$ and the backward wave $w^-(s,t)$ along $t^-(\sigma;s,t)$. Without details - the reader is sent to [18] - the following system of coupled delay differential and difference equations is associated

$$\begin{aligned} &m_o \frac{\mathrm{d}v_H}{\mathrm{d}t} + \left(\gamma\text{'}_o + \frac{c_a \gamma\text{'}_l}{E\Gamma}\right) v_H + \gamma\text{'}_l y^-(t - L/c_a) = f_H(t) \\ &m_b \frac{\mathrm{d}v_b}{\mathrm{d}t} + \frac{E\Gamma}{c_a} v_b - \frac{2E\Gamma}{c_a} y^+(t - L/c_a) = 2f_b(t) \\ &y^+(t) = y^-(t - L/c_a) + \frac{2c_a \gamma\text{'}_l}{E\Gamma} v_H(t) \\ &y^-(t) = -y^+(t - L/c_a) + v_b(t) \end{aligned} \quad (23)$$

and the following basic result is true

Theorem 1: Consider the initial boundary value problem (19), having $v_H(0)$, $v_o(s)$, $w_o(s)$, $0 \le s \le L$, a set of sufficiently smooth initial conditions, the Riemann invariants (20)-(21) and the initial boundary value problem (22) with the initial conditions $v_H(0)$, $w^{\pm}(s,0) = v_o(s) \mp c_a w_o(s)$, $0 \le s \le L$. Let $(v_H(t), w^{\pm}(s,t))$ be a classical solution of (22) and define the functions

$$u^+(t) := w^+(L,t) \;,\; u^-(t) := w^-(0,t) \;;\; y^{\pm}(t) := u^{\pm}(t + L/c_a) \quad (24)$$

Then the functions $(v_H(t), v_b(t), y^{\pm}(t))$ are a (possibly discontinuous) solution of (23) defined by the initial conditions

$$\begin{aligned} &v_H(0) \;,\; v_b(0) = v_o(L) \;,\; y_o^+(t) = v_o(-c_a t) - w_o(-c_a t) \\ &y_o^-(t) = v_o(L + c_a t) + w_o(L + c_a t) \;,\; -L/c_a \le t \le 0 \end{aligned} \quad (25)$$

Conversely, let $(v_H(t), v_b(t), y^{\pm}(t))$ be a (possibly discontinuous) solution of (23) defined by some initial conditions. Then the functions $(v_H(t), v(s,t), w(s,t))$ where $v(s,t)$ and $w(s,t)$ are defined by (21) and $w^{\pm}(s,t)$ by the representation formulae

$$w^+(s,t) = y^+(t - s/c_a) \;,\; w^-(s,t) = y^-(t + (s-L)/c_a) \quad (26)$$

are a classical, possibly discontinuous solution of (19) with the initial conditions resulting accordingly.

The proof of this theorem can be done following the methodology of [18] - the proof of Theorem 3.1 - and we shall omit it. Its significance is however rather important: it states a one to one correspondence between the solutions of two mathematical objects: the initial boundary value problem (19) (or (22)) and the system of functional equations (coupled delay differential and difference) (23). Consequently all results obtained for one of them are automatically projected back on the other.

III.5 The basic theory. Now, system (23) belongs to the class of functional differential equations of neutral type. The simplest proof of this assertion is to substitute $v_H(t)$ and $v_b(t)$ from the difference equations into the differential ones and obtain a neutral system. But (23) itself is of neutral type according to the by now classical reference [16]. Also the fact that its solutions are neither smoothed in time nor loosing their smoothness puts system (23) in the class of neutral systems according to the classification of G.A. Kamenskii [19].

The construction of the solution of (23) can be done by steps: the solution has the smoothness of its initial conditions and discontinuities at $t = kL/c_a$, $k = 0, \pm 1, \pm 2, \ldots$ (as for any system of neutral type the solution can be constructed forward and backward also). Being a linear system, uniqueness and continuous data dependence are ensured.

Conversely, the representation formulae (26) will ensure the same properties for the discontinuous classical solution of (18). Moreover, if one takes into account a result of Hale concerning the theory of neutral functional differential equations on Sobolev spaces, the way to the generalized solutions [14] is thus open. We end here the discussion concerning the basic theory.

IV. Inherent stability of the system of axial vibrations

Some time ago there was introduced the concept of *model augmented validation* [18]. Following the line of the *Stability Postulate* of N. G. Četaev [22] it was stressed in the aforementioned cited papers that inherent stability should be considered a distinct and necessary step in validating a mathematical model, together with basic theory (well posedness in the sense of J. Hadamard).

IV.1 Inherent stability. We consider here stability of (23) with $f_H(t) \equiv f_b(t) \equiv 0$. This autonomous system of neutral type is in a critical case: the basic assumption of strong stability for the difference operator *does not hold*; the matrix D defining this operator has $\pm \imath$ as eigenvalues. If however the time constant $m_o(\gamma\text{'}_o + (c_a\gamma\text{'}_l)(E\Gamma)^{-1})^{-1}$ can be considered small with respect to e.g. the time constant $m_b c_a (E\Gamma)^{-1}$, then this fast dynamics can be neglected (relying on singular perturbations) to obtain the reduced system of the slow dynamics

$$\begin{aligned} & m_b \frac{\mathrm{d}v_b}{\mathrm{d}t} + \frac{E\Gamma}{c_a} v_b - \frac{2E\Gamma}{c_a} y^+(t - L/c_a) = 0 \\ & y^+(t) = \frac{\gamma\text{'}_0 E\Gamma + c_a\gamma\text{'}_l - 2c_a(\gamma\text{'}_l)^2}{\gamma\text{'}_0 E\Gamma + c_a\gamma\text{'}_l} y^-(t - L/c_a) \\ & y^-(t) = -y^+(t - L/c_a) + v_b(t) \end{aligned} \tag{27}$$

The difference operator of (27) will be strongly stable if the coefficient ρ of $y^-(t - L/c_a)$ in the second equation will satisfy $|\rho| < 1$ and this condition holds iff

$$0 \leq \gamma\text{'}_l < \frac{1}{2}\left(1 + \sqrt{1 + 4\gamma\text{'}_o \frac{E\Gamma}{c_a}}\right) \tag{28}$$

We shall thus discuss first stability of (27) whose characteristic equation is

$$p(\lambda) = (c_a m_b \lambda + E\Gamma)\mathrm{e}^{2\lambda L/c_a} + \rho(c_a m_b \lambda - E\Gamma) \tag{29}$$

Defining some elementary changes of variables in (29) the stability problem is reduced to the location of the roots of the quasi-polynomial

$$\begin{aligned} q(z) = & \left(z + \frac{E\Gamma L}{c_a^2 m_b} \cdot \frac{1 - \rho\mathrm{e}^{\alpha}}{1 + \rho\mathrm{e}^{\alpha}} - \frac{\alpha}{2}\right)\cosh z + \\ & + \left(\frac{1 - \rho\mathrm{e}^{\alpha}}{1 + \rho\mathrm{e}^{\alpha}} z + \frac{E\Gamma L}{c_a^2 m_b} - \frac{\alpha}{2} \cdot \frac{1 - \rho\mathrm{e}^{\alpha}}{1 + \rho\mathrm{e}^{\alpha}}\right)\sinh z \end{aligned} \tag{30}$$

in the left half plane $\Re e(z) < 0$ for $\alpha > 0$ sufficiently small. This will give the roots of (29) in $\Re e(\lambda) \leq -\alpha c_a/2L$. The approach to be used is the sharpest one, based on the generalized Sturm method [23]. Firstly we shall consider the generalized necessary conditions of Stodola type - see [23], page 264, then the necessary and sufficient conditions resulting from the analysis of the Sturm sequence. It turns out quite easily that by choosing $\alpha > 0$ sufficiently small to have

$$\alpha \cdot \frac{1 + \rho\mathrm{e}^{\alpha}}{1 - \rho\mathrm{e}^{\alpha}} < \frac{2E\Gamma L}{c_a^2 m_b} \tag{31}$$

then the roots will result in $\Re e(z) < 0$ - see [23], page 296.

The aforementioned result, ensuring location of the roots of (29) in $\Re e(\lambda) \leq -\alpha c_a/2L$, suggests examination of the critical case of (23) based on singular perturbations i.e. for $m_o > 0$ sufficiently small.

IV.2 Inherent stability in the critical case. We consider now the characteristic quasi-polynomial of (23)

$$\begin{aligned} p(\lambda) = & \left(m_o\lambda + \gamma\text{'}_o + \frac{c_a\gamma\text{'}_l}{E\Gamma}\right)\left(m_b\lambda + \frac{E\Gamma}{c_a}\right) - \\ & - \left(m_o\lambda + \gamma\text{'}_o + \frac{c_a\gamma\text{'}_l}{E\Gamma}(1 - \gamma\text{'}_l)\right)\left(m_b\lambda - \frac{E\Gamma}{c_a}\right)\mathrm{e}^{-2\lambda L/c_a} \end{aligned} \tag{32}$$

Let $\lambda = s - \alpha$, $\alpha > 0$ hence $\Re e(s) < 0 \Leftrightarrow \Re e(\lambda) \leq -\alpha$. We shall have

$$\begin{aligned} \mathrm{e}^{(s-\alpha)L/c_a} p(s - \alpha) = & \left[-2m_o m_b\left(\sinh\frac{\alpha L}{c_a}\right)s^2 + \ldots\right]\cosh\frac{sL}{c_a} \\ & + \left[2m_o m_b\left(\cosh\frac{\alpha L}{c_a}\right)s^2 + \ldots\right]\sinh\frac{sL}{c_a} \end{aligned}$$

and the two coefficients of the quadratic terms have opposite signs. Therefore $p(s - \alpha)$ cannot have its roots in $\Re e(s) < 0$ - see [23], page 264. It follows that there is no $\alpha > 0$ such that the roots of (32) lie in $\Re e(\lambda) \leq -\alpha$. On the other hand by taking $\alpha < 0$ it is not difficult to find that (32) has no roots the right half plane.

We check now if the roots of (32) lie at least in $\Re e(\lambda) < 0$. Applying the Sturm method - see again [23] - we find that (32) is in a limit (marginal) case ([23], page 288) hence it can have some roots on the imaginary axis.

At the same time the asymptotic properties of the quasi-polynomials arising from neutral equations in the critical case of the difference operator show that even some chains of roots in the left half plane can accumulate asymptotically to the imaginary axis without crossing it - see [24] or [25]. A discussion of such cases can be found within the "Supplementary remarks" at Chapter 9 of [26]; it points out to non-uniform asymptotic stability in the best case.

V. Some conclusions and perspective problems

The underlying philosophy of this paper is that control of some physical phenomena displayed by an industry device can ensure improved performance if it relies on a sound mathematical model. For this reason the problem of drilling vibration modeling - for both torsional and axial vibrations

- has been tackled in all its generality. More specific, the modeling was considered within the variational approach of the Hamiltonian Mechanics. It became quite clear that the quality of the model strongly depends on the *"list"* of external forces and momenta included in the model of the mechanical work. The fact that the equations for the torsional and axial vibrations are quite decoupled and can be treated separately is due to the chosen model of the bit-rock friction force and/or momentum.

The paper focused on the axial vibrations whose model resulted linear with external control and perturbation signals. The rather complete list of forces and momenta led to a model that displayed a critical case of stability. To be more specific, in the paper there has been used the approach of integrating the Riemann invariants along the characteristics and associating a system of functional differential equations of neutral type. The difference operator of the system of neutral type results only marginally (critically) stable instead of strongly stable. Such *"pathologies"* have been discussed long ago [19] and easily forgotten because not reported in physics, engineering and other applications. Now we were facing exactly an application of this kind. Neglecting a (presumably) small parameter (not present in earlier models) the difference operator became strongly stable and the model got exponential stability. A possible use of *singular perturbations theory* is thus suggested.

According to our opinion, the main perspective problems connected to the drilling vibrations are thus of mathematical nature. Criticality of the difference operator induces in fact hidden oscillations due to weakly damped modes: such modes are associated to those roots of the characteristic equation which accumulate asymptotically to the imaginary axis while being located in the left half plane. They define probably a series [27] which is not uniformly convergent. If the frequencies associated to it define an increasing sequence (our *conjecture*) then they might be quenched by the feedback control provided the closed loop bandwidth is narrow enough (another *conjecture* of ours). Finally the *third conjecture* might be connected to the role of the neglected internal distributed damping of the drillstring: the associated functional equations might display a strongly stable difference operator.

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References

[1] M. S. Márquez, I. Boussaada, H. Mounier, and S. I. Niculescu, *Analysis and Control of Oilwell Drilling Vibrations. A Time-Delay Systems Approach*, ser. Advances in Industrial Control. Berlin. Basel. London: Springer, 2015.

[2] M. B. Saldivar, "Analysis, modeling and control of an oilwell drilling system," Ph.D. dissertation, Centro de Investigacin y de Estudios Avanzados del Instituto Politcnico Nacional, Mexico. Institut de Recherche en Communications et Cyberntique de Nantes, France, 2013.

[3] M. A. Kiseleva, "Oscillations and stability of drilling systems: Analytical and numerical methods," Ph.D. dissertation, St. Petersburg State University. School of Mathematics and Mechanics Russia, 2013.

[4] D. Bresch-Pietri and M. Krstić, "Output-feedback adaptive control of a wave pde with boundary anti-damping," *Automatica*, vol. 50, pp. 1407–1415, 2014.

[5] A. Serrarens, M. van-de Molengraft, and L. van-den Steen, "h_∞ control for suppressing stick-slip in oil well drillstrings," *IEEE Control Systems*, vol. 18(2), pp. 19–30, 1998.

[6] C. Canudas de Wit, F. R. Rubio, and M. A. Corchero, "D-oskil: A new mechanism for controlling stick-slip oscillations in oil well drillstrings," *IEEE Transactions on Control Systems Technology*, vol. 16(6), pp. 1177–1191, 2008.

[7] M. A. Kiseleva, N. V. Kondratyeva, N. V. Kuznetsov, and G. A. Leonov, "Hidden oscillations in drilling systems with salient pole synchronous motor," *IFAC PapersOnLine*, vol. 48-11, pp. 700–705, 2015.

[8] H. Goldstein, C. P. Poole, and J. Safko, *Classical Mechanics*. London: Pearson Addison Wesley, 2011.

[9] L. Meirovitch, *Elements of Vibration Analysis*. New York: McGraw-Hill, 1975.

[10] ——, *Methods of Analytical Dynamics*. Mineola New York: Dover Publications, 1998.

[11] N. Akhiezer, *Variational calculus (in Russian)*. Kharkov (USSR): "Visča Škola" Publ. House, 1981.

[12] B. Saldivar, I. Boussaada, H. Mounier, S. Mondié, and S. I. Niculescu, "An overview on the modeling of oilwell drilling vibrations," *IFAC Proceedings Volumes*, vol. 47-3, pp. 5169–5174, 2014.

[13] B. Saldivar, S. Mondié, S.-I. Niculescu, H. Mounier, , and I. Boussaada, "A control oriented guided tour in oilwell drilling vibration modeling," *Annual Reviews in Control*, vol. 42, pp. 100–113, 2016.

[14] V. E. Abolinia and A. D. Myshkis, "Mixed problem for an almost linear hyperbolic system in the plane (russian)," *Mat. Sbornik*, vol. 50:92, pp. 423–442, 1960.

[15] K. L. Cooke and D. W. Krumme, "Differential-difference equations and nonlinear initial-boundary value problems for linear hyperbolic partial differential equations," *Journal of Mathematical Analysis and Applications*, vol. 24, pp. 372–387, 1968.

[16] K. L. Cooke, "A linear mixed problem with derivative boundary conditions," in *Seminar on Differential Equations and Dynamical Systems (III)*, ser. Lecture Series, D. Sweet and J. A. Yorke, Eds. College Park: University of Maryland, 1970, vol. 51, pp. 11–17.

[17] V. Răsvan, "A method for distributed parameter control systems and electrical networks analysis," *Rev. Roumaine Sci. Techn. Série Electrotechn. Energ*, vol. 20, pp. 561–566, 1975.

[18] ——, "Augmented validation and a stabilization approach for systems with propagation," in *Systems Theory: Perspectives, Applications and Developments*, F. Miranda, Ed. New York: Nova Science Publishers, 2014, pp. 125–169.

[19] L. E. El'sgol'ts and S. B. Norkin, *Introduction to the theory of differential equations with deviated argument (in Russian)*. Moscow (USSR): "Nauka" Publ. House, 1971.

[20] M. A. Cruz and H. J. K., "Stability of functional differential equations of neutral type," *J. Differ. Equations*, vol. 7, pp. 334–355, 1970.

[21] D. L. Russell, "Mathematical models for elastic beams and their control-theoretic implications," in *Semi-Groups: Theory and Applications*, H. Brézis, M. G. Crandall, and F. Kappel, Eds. Essex: Longman, 1986, vol. 2, pp. 177–216.

[22] V. Răsvan, "The stability postulate of n. g. Četaev," *IFAC PapersOnLine*, vol. 50(1), pp. 7450–7455, 2017.

[23] N. G. Čebotarev and N. N. Meiman, "The routh-hurwitz problem for polynomials and entire functions," *Proc. Mat. Inst. Steklov*, vol. 26, pp. 3–331, 1949.

[24] R. E. Bellman and K. L. Cooke, *Differential Difference Equations*, ser. Mathematics in Science and Engineering. New York Toronto London: Academic Press, 1963, vol. 6.

[25] A. M. Zverkin, "Series expansions of solutions of differential difference equations. i: quasi-polynomials (in russian)," *Proc. Seminar Theory of differential equations with deviated argument*, vol. 3, pp. 3–38, 1965.

[26] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, ser. Applied Mathematical Sciences. New York Berlin: Springer, 1993, vol. 99.

[27] A. M. Zverkin, "Series expansions of solutions of differential difference equations. ii: series expansions (in russian)," *Proc. Seminar Theory of differential equations with deviated argument*, vol. 4, pp. 3–50, 1967.